A NOTE ON DIVIDING ANGLES TO EQUAL PARTS

Reza HABIBI[1]
[1] Iran Banking Institute, Department of Banking, Tehran (Iran)
r_habibi@ibi.ac.ir

Abstract

Dividing angles to three equal parts is a traditional problem in geometry. Some mathematician has proved that it is impossible using the usual tools in geometry. By the way, some approximated methods are given. In this paper, we give a simple recursive method. We show that our method works well, and its convergence rate is good. This method enables us, to divide an angle with unknown size to k > 2 equal parts. The stochastic version of our method is also considered.

Keywords: Angle, Geometry, Rate of convergence, Recursive partitions

1. Introduction

An angle is a figure in geometry which its position, direction, precision are too important in real world. It forms when two rays meet at a common vertex. An angle can be divided to two equal parts by angle bisector. However, it is a very old problem in geometry that it is impossible to divide an angle with unknown size to 3 equal parts using the straightedge and compasses. The routine method for bisection of geometric angles has continued to dominate the scene despite the fact that it divides angles only into 2, 4, 8 etc. excluding 3, 5, 6, 7, 9 etc. equal parts. This could be caused by the absence of other simple methods or that some angles that cannot be obtained through bisection can be copied from the protractor or the use of set squares, see Elekwa (2011). The method by Odogwu (2015) is mathematically complex, tedious and cannot be practiced easily. There is therefore the need for other methods to be introduced.

In this paper, we show that it is possible to divide an angel to $k > 2$ equal parts using algebraic infinite series. Dividing an angle with size α, (α unknown) is equivalent to dividing a line with length α. To see that, it is enough to consider the angle, as a central angle of a circle with unit radius.

1.1. Three parts.

It is easy to see that (see for example Rudin, 1976)

$$\sum_{i=0}^{\infty}(-1/2)^i = \frac{2}{3}.$$

Here, we interpret this series, geometrically. Consider line A_0A_1 with (unknown) length α. Suppose that a particle is at A_1. Particle moves from A_1 to A_0. From A_0, the particle returns to the middle of A_0A_1 (call it A_2). Note that it isn't necessary to know the size of lines (angels) for dividing them to two equal parts. Our moving particle goes from A_2 to A_3 (the middle of A_0A_2) and returns to A_4. Repeat this moving and returning until infinity. Figure 1 illustrates more.

ISSN 2694-8133

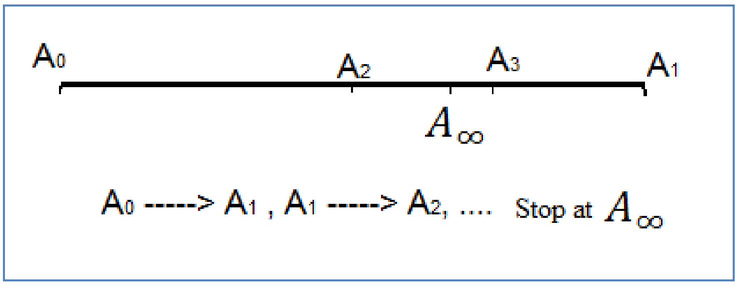

Figure 1. Particle movements among A's

It is obvious the particle stops at A_∞. Since the length of $A_n A_{n-1}$ converges to zero. The length of $A_1 A_\infty$ equals to distance moved by particle which is

$$\alpha(1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+...) = \alpha\sum_{i=0}^{\infty}(-1/2)^i = \frac{2}{3}\alpha .$$

Therefore, it is enough to divide $A_1 A_\infty$ to two equal parts. In fact, we have proved our claim. It is easy to see that

$$|\alpha\sum_{i=0}^{n}(-1/2)^i - \frac{2}{3}\alpha| = (\alpha/3)2^{-n} .$$

This shows that the speed of convergence A_n to A_∞ (that is 2^{-n}) is good. The above equation also suggests to drop the first *n* moves and to start from A_n. As follows, a simple proof is given for $\sum_{i=0}^{\infty}(-1/2)^i = \frac{2}{3}$. Let *n* be an arbitrary natural number and *a* be a real number such that $|a| < 1$. Then, $\sum_{i=1}^{n}(-a)^i = \frac{1-(-a)^{n+1}}{1+a}$. As $n \to \infty$, then $(-a)^{n+1}$ goes to zero and $\sum_{i=1}^{\infty}(-a)^i = \frac{1}{1+a}$. By assuming $a = 0.5$, the proof is completed.

1.2. The *k* parts

In this part, we generalize our result to *k* equal parts. Our approach is based on induction. We show that if we can divide a line to *k* > 2 parts then partitioning to (*k* + 1) parts is possible. In this way, since we have done for *k* = 3, therefore we have done for *k* = 3, 4, ... The proof is easy. One can check that

$$\sum_{i=0}^{\infty}(-1/k)^i = \frac{k}{k+1}.$$

Again, consider moving particle and in each stage divide the segmented line to *k* parts (instead of *2* parts). The above infinite series says that you will divide the $A_0 A_1$ to *k* equal parts.

2. Generalizations

Here, we rephrase our problem. Suppose that *f* is positive function such that $f(\alpha) \le \alpha$. How can we part $f(\alpha)$ from line (angel) with unknown length α. In the previous section, we solve this problem for $f(\alpha) = \alpha / k$. Here, we present some special cases. Let $f(\alpha) = \frac{\alpha}{r}$ where $r \ge 1$ is a rational number. Let $r = \frac{k'}{k''}$ for some $1 = k' \le k'' = 2,3,...$. Then $f(\alpha) = \frac{k''}{k'}\alpha$.

Therefore, the problem reduces to deriving $\frac{1}{k'}$ of an angle with size $k''\alpha$. This is solved in the previous section. A natural generalization is $f(\alpha)=c\alpha$ where $0<c\leq 1$ is an irrational number. There is a sequence of rational numbers $\{c_n\}$ such that $c_n \uparrow c$ (Rudin, 1976). Let $d_n = c_n - c_{n-1}$ with $c_0 = 0$. Then

$$d_n = \sum_{i=1}^{n} d_i .$$

To update the moving and returning algorithm, in the *n-th* stages of algorithm, let particle moves from A_{n-1} to A_n such that the length of $A_n A_{n-1}$ is d_n Therefore, we have shown we can solve our problem for $f(\alpha)=b\alpha$ where $0<b\leq 1$ is a positive real number.

3. Stochastic version

In this section, we consider the stochastic version of above mentioned problem. Suppose that $U_1,...,U_n$ are independent and identically distributed random variables defined on (0,1). Consider again the line A_0A_1 and mentioned moving particle. Suppose that the moving particle goes from A_0 to A_1. Then, it returns to A_2 where the length of A_1A_2 is $U_1\alpha$. That is, the particle returns $U_1\alpha$. units. This possibility is shown in previous section.
Next, the particle moves form A_2 to A_3 such that the length of A_2A_3 is $U_2U_1\alpha$. By letting $U_1 = U_2 = ... = 1/2$ with probability one, the angle is divided to 3 equal parts. The location of particle in *n*-th steps is given by

$$S_n = 1 + \sum_{k=1}^{n} (-1)^k \prod_{i=1}^{k} U_i \ .$$

Suppose that the expectation of U_1 is μ. One can show that

$$E(S_n) = 1 + \sum_{k=1}^{n} (-\mu)^k = 1 - \frac{1-(-\mu)^n}{1+\mu}\mu.$$

Since $0<\mu<1$, then $|(-\mu)^n|$ goes to zero as *n* goes to infinity. Using the Monte Carlo method, then, $E(S_n)$ converges to $\frac{1}{1+\mu}$. As soon as faced with uncertainty in estimation of unknown parameter, Monte Carlo method uses multiple values and averages the results.
If we suppose that U_1 is uniformly distributed on (0,1) then $E(S_n)$ converges to 2/3. In this way, the angle is divided to 3 equal parts. If we suppose that U_1 has a beta distribution with parameter α, β then the limiting values of $E(S_n)$ is $\frac{\theta}{\alpha+\theta}$, where $\theta = \alpha + \beta$. By choosing suitable values for α, β the angle can be divided every rational number. The rate of convergence of $E(S_n)$ to $\frac{1}{1+\mu}$ is $(-\mu)^n$.

Next, suppose that for R realization of U_i's, $i = 1, \ldots, n$, we derive the $S_n^1, \ldots, S_n^R$, independently. The SLLN (Resnick, 2001) guaranties that $\overline{S_R} = (1/R)\sum_{r=1}^{R} S_n^r$ converges almost surely to $\frac{1}{1+\mu}$ as n, R goes to infinity. The following Table gives the values of $\overline{S_R}$ for some selected values of n and R=1000, under the uniform distribution, and the corresponding standard deviations.

Table 1. Values of $\overline{S_R}$

n	10	25	40	70	100
$\overline{S_R}$	0.577	0.5633	0.5258	0.5087	0.5012
$stdev(\overline{S_R})$	0.00758	0.00728	0.007547	0.00731	0.00727

4. Simulation.

In this section, it is surveyed, how simulation methods such as Monte Carlo approach may be applied to divide an angle with parts in size of an arbitrary function of angle. To this end, first, notice the following Remark.

Remark 1. The main idea of previous section was the use of Taylor expansion $f(x) = \frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$. In mathematics, taylor expansion of a function is an infinite sum of terms that is expressed in terms of function dreivatives at a single point (Rudin, 1976). Solomon (1991) proposed a simulation based method for approximation of infinite series. Here, first, his method is applied to $\sum_{\mathrm{i}=0}^{\infty}\left(\frac{1}{2}\right)^{\mathrm{i}}$ and second it is extended for general functions. Then, a geometrical method to divide an angle to three equal parts is given. First notice that $\sum_{\mathrm{i}=1}^{\infty}\left(\frac{1}{2}\right)^{\mathrm{i}} = 1$ and $\sum_{\mathrm{i}=1}^{\infty}(-1)^{\mathrm{i}+1}2^{-\mathrm{i}} = \frac{1}{3}$. Therefore, $\frac{1}{3} = \mathrm{E}\,(-1)^{\mathrm{N}+1}$ where $\mathrm{P}(\mathrm{N} = \mathrm{i}) = 2^{-\mathrm{i}}, \mathrm{i} \geq 1$. Notations E and P stand for mathematical expectations and probability functions, respectively. Let $\mathrm{U_n}$'s be a sequence of independent and identically uniformly distributed random variables on (0,1) and N is the first index such that $\mathrm{U_N}$ gets larger than 0.5. Then, $\mathrm{P}(\mathrm{N} = \mathrm{i}) = 2^{-\mathrm{i}}, \mathrm{i} \geq 1$. Therefore, it is enough to generate uniform random variables and then find the first index such that it goes beyond 0.5 and to find the average of $(-1)^{\mathrm{N}+1}$.

For general functions which is differentiable infinitely times, the Taylor expansion if given by

$$\boldsymbol{f(\alpha) = f(\alpha_0) + \sum_{i=1}^{\infty} f^{(i)}(\alpha_0)(\alpha - \alpha_0)^i / i! =}$$
$$\boldsymbol{= f(\alpha_0) + \sum_{i=1}^{\infty} p_i\, a_i},$$

where $\boldsymbol{p_i = \frac{i-1}{i!}}$ and $\boldsymbol{a_i = \frac{f^{(i)}(\alpha_0)(\alpha - \alpha_0)^i}{(i-1)}}$.

Then, $\boldsymbol{f(\alpha) = f(\alpha_0) + E(a_N)}$. For example, for $\boldsymbol{f(\alpha) = e^{-\alpha}}$, then $\boldsymbol{f(\alpha) = 1 + E(a_N)}$, where

$$\boldsymbol{\alpha_0 = 0,\ a_i = \frac{(-\alpha)^i}{(i-1)}}.$$

Following Solomon (1991) the point N is the first index at which $\boldsymbol{U_N > U_{N-1}}$. The following algorithm gives the geometrical approach, briefly.

(a) Consider interior point $\boldsymbol{\alpha_0}$ from support f and derive the taylor expansion.

(b) Move for $\boldsymbol{f(\alpha_0)}$ and move in size of $\boldsymbol{a_i = \frac{f^{(i)}(\alpha_0)(\alpha - \alpha_0)^i}{(i-1)}}$ with probability of $\boldsymbol{p_i = P(N = i) = \frac{i-1}{i!}}$. Obtain m times of point N and take sample average of $\boldsymbol{a_i}$'s.

(c) To simulate N, notice that N is the first index at which $\boldsymbol{U_N > U_{N-1}}$.

5. Conclusion

This paper studied the method for dividing an angle to equal parts. First, methods for dividing angles to three equal parts are studied. After then, it is verified, how for a general function of angle is done. Simulation methods and Monte Carlo computations play important role in this part.

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