THE SHORTEST KNIGHT'S TOURS

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Abstract

The knight's tour on an 8×8 chessboard is one of the best-known mathematical chess problems, and it is attractive also for pupils aged 6 to 11. This problem has been resolved generally for the chessboard $m \times n$; thus, the question arises as to which knight's tours are the shortest. These knight's tours can be interesting even for younger students because they won't spend too much time trying to solve them. We will study both open and closed knight's tours.

Keywords: Mathematical chess problems, combinatorics, knight's tour.

1. Introduction

In our paper, we will deal with the shortest knight's tours. The knight's tour is a mathematical chess problem. We recall ("Mathematical chess problems", Wikipedia) that a mathematical chess problem is a mathematical problem that is formulated using a chessboard and chess pieces.

The knight's tour problem asks for a tour of a knight who visits all squares on a chess board ("Mathematical chess problems", Wikipedia) exactly once. If the knight's tour ends on a square from which we can get to the first square of the knight's tour by a knight's move, we speak of a closed knight's tour. Otherwise, we speak of an open knight's tour ("Knight's tours", Wikipedia).

A knight moves two squares horizontally, one square vertically, or one square horizontally, then two vertically. For example, in Figure 1, a white knight on the square h2 can move on the selected squares g4, f3, and f1. In Figure 2, a white knight on the square a5 can move on the selected squares b3, b7, c4, and c6.

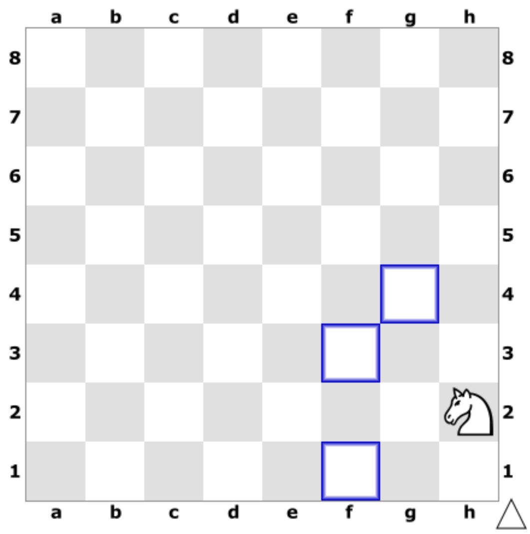

Figure 1. Moves of a knight on h2.

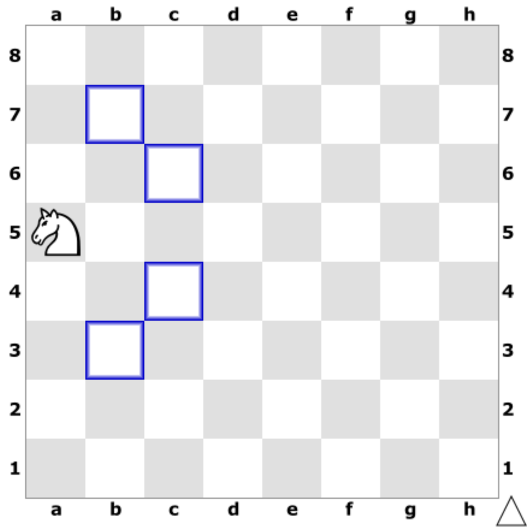

Figure 2. Moves of a knight on a5.

The first mentions of knight's tours date from the 9th century AD ("Knight's tours", Wikipedia). Each closed knight's tour can be done in two directions. If we consider these two walks differing only in direction as a single walk, then the total number of possible closed walks on the 8×8 board is 13 267 364 410 532 ("Knight's tours", Wikipedia). Among the most famous knight's tours is Euler's knight's tour from 1759, which seems to be attractive also for pupils aged 6 to 11 since her algorithm can use four colors (Pastor, 2023); see Figure 3 for an example of Euler's closed knight's tour and Figure 4 for an example of Euler's open knight's tour.

51	38	15	32	63	36	1	28
14	17	50	37	2	29	62	35
39	52	31	16	33	64	27	4
18	13	40	49	30	3	34	61
53	42	9	22	57	48	5	26
12	19	56	41	8	23	60	47
43	54	21	10	45	58	25	6
20	11	44	55	24	7	46	59

Figure 3. Euler's closed knight's tour.

The problem of a knight's tour on the classic 8×8 chessboard can be generalized to the knight's tour on any rectangular chessboard. It is well known for which chessboards the open and closed knight's tour, respectively, is feasible. The relevant theorems are recalled in Sections 2 and 3, respectively. Our article will focus on which chessboard the knight's tours are the shortest. These knight's tours can be interesting even for pupils aged 6 to 11 because they won't spend too much time trying to solve them. We will deal with both closed and open knight's tours.

1	18	61	46	15	32	59	34
62	47	2	17	60	35	14	31
19	4	45	64	29	16	33	58
48	63	20	3	36	57	30	13
5	22	49	44	9	28	55	38
50	43	8	21	56	37	12	27
23	6	41	52	25	10	39	54
42	51	24	7	40	53	26	11

Figure 4. Euler's open knight's tour.

2. The shortest closed knight's tour

In his relatively short article, A.L. Schwenk showed (Schwenk, 1991) for which rectangular chessboards it is possible to realize a closed knight's tour.

Theorem 1. An $m \times n$ chessboard with $m \leq n$ has a closed knight's tour unless one or more of these three conditions holds:

i. m and n are both odd,
ii. m = 1, 2, or 4,
iii. m = 3 and n = 4, 6, or 8.

We say that a chessboard $m \times n$ has a measure $m \cdot n$, i.e., the measure of the chessboard equals to the number of its squares. The knight's tour on a chessboard A is shorter than the knight's tour on a chessboard B if the measure of A is less than that of B.

Table 1 shows on which chessboards $m \times n$, $3 \leq m \leq 6$, $m \leq n$, $3 \leq n \leq 15$, it is possible to realize a closed knight's tour. The green color of the corresponding square in Table 1 means that the corresponding closed knight's tour is possible. The red color means that the corresponding closed knight's tour is impossible. In the case of admissible closed knight's tours, the measure of the considered chessboard is given.

Table 1. The chessboards and the closed knight's tours.

	m = 3	m = 4	m = 5	m = 6
n = 3		x	x	x
n = 4			x	x
n = 5				x
n = 6			30	36
n = 7				42
n = 8			40	48
n = 9				54
n = 10	30		50	60
n = 11				66
n = 12	36		60	72
n = 13				78
n = 14	42		70	84
n = 15				90

Analyzing Table 1, we can easily find the order of chessboards according to their measure on which it is possible to realize a closed knight's tour, see Table 2.

Table 2. The shortest closed knight's tour.

order	**chessboards**	**measure**
1.–2.	3 × 10	30
1.–2.	5 × 6	30
3.–4.	3 × 12	36
3.–4.	6 × 6	36
5.	5 × 8	40

There are precisely six closed knight's tours on the chessboard 3 × 10 (Knight´s Tour Notes, 2023) if we again consider two walks differing only in direction as a single walk. Figure 5 shows one of them.

29	2	5	14	11	26	9	18	23	20
4	13	30	27	6	15	24	21	8	17
1	28	3	12	25	10	7	16	19	22

Figure 5. The closed knight's tour 3 × 10.

On the chessboard 5 × 6, there exist only three closed knight's tours. Figure 6 shows the tour found by Euler in 1759 (Knight´s Tour Notes, 2023).

1	14	25	6	29	16
24	5	30	15	20	7
13	2	11	26	17	28
10	23	4	19	8	21
3	12	9	22	27	18

Figure 6. The closed knight's tour 5 × 6.

The chessboard 3 × 12 offers 44 closed knight's tours; let us show one of them in Figure 7 (Knight´s Tour Notes, 2023).

35	2	33	8	5	28	17	14	11	26	23	20
32	7	36	3	30	15	10	27	18	21	12	25
1	34	31	6	9	4	29	16	13	24	19	22

Figure 7. The closed knight's tour 3 × 12.

There are 1245 closed knight's tours on the chessboard 6 × 6; Figure 8 shows one of them (Knight´s Tour Notes, 2023).

32	7	20	17	30	5
21	18	31	6	27	16
8	33	10	19	4	29
11	22	1	28	15	26
34	9	24	13	36	3
23	12	35	2	25	14

Figure 8. The closed knight's tour 6 × 6.

Finally, on the chessboard 5 × 8, there exist 11 closed knight's tours; see Figure 9 for one of them (Knight´s Tour Notes, 2023).

22	7	40	5	20	15	30	33
39	4	21	8	29	32	19	14
26	23	6	1	16	11	34	31
3	38	25	28	9	36	13	18
24	27	2	37	12	17	10	35

Figure 9. The closed knight's tour 5 × 8.

3. The shortest open knight's tour

Following (Conrad et al., 1994) and (Knight´s Tour Notes, 2023), we can get the answer to the question of which rectangular chessboards it is possible to realize an open knight's tour; see also (Wikipedia, 2024b).

Theorem 2. An $m \times n$ chessboard with $m \leq n$ has an open knight's tour unless one or more of these three conditions holds:

i. $m = 1$ or 2,
ii. $m = 3$ and $n = 3$, 5, or 6,
iii. $m = 4$ and $n = 4$.

Table 3 shows on which chessboards $m \times n$, $3 \leq m \leq 6$, $m \leq n$, $3 \leq n \leq 15$, it is possible to realize a closed knight's tour. The meaning of the colors is the same as in Table 1.

Table 3. The chessboards and the open knight's tours.

	$m = 3$	$m = 4$	$m = 5$	$m = 6$
$n = 3$		x	x	x
$n = 4$	12		x	x
$n = 5$		20	25	x
$n = 6$		24	30	36
$n = 7$	21	28	35	42
$n = 8$	24	32	40	48
$n = 9$	27	36	45	54
$n = 10$	30	40	50	60
$n = 11$	33	44	55	66
$n = 12$	36	48	60	72
$n = 13$	39	52	65	78
$n = 14$	42	56	70	84
$n = 15$	45	60	75	90

Analyzing Table 3, we can easily find the order of chessboards according to their measure on which it is possible to realize a closed knight's tour, see Table 4.

Table 4. The shortest open knight's tour.

order	**chessboards**	**measure**
1.	3×4	12
2.	4×5	20
3.	3×7	21
4.–5.	3×8	24
4.–5.	4×6	24

Figures 10–14 show examples of the shortest open knight's tours; for more details, see (Knight´s Tour Notes, 2023). Let us note only that there exist 8 open knight's tours on the chessboard 3×4 and 744 on the chessboard 4×6 if we again consider two walks differing only in direction as a single walk.

1	4	7	10
8	11	2	5
3	6	9	12

Figure 10. The open knight's tour 3×4.

1	20	7	16	3
6	15	2	11	8
19	10	13	4	17
14	5	18	9	12

Figure 11. The open knight's tour 4×5.

1	4	7	18	15	10	13
6	21	2	9	12	19	16
3	8	5	20	17	14	11

Figure 12. The open knight's tour 3 × 7.

3	6	9	12	15	18	23	20
8	11	2	5	24	21	14	17
1	4	7	10	13	16	19	22

Figure 13. The open knight's tour 3 × 8.

14	1	16	5	24	3
17	10	13	2	21	6
12	15	8	19	4	23
9	18	11	22	7	20

Figure 14. The open knight's tour 4 × 6.

4. Conclusion

An open knight's tour on smaller-sized chessboards can be interesting for children aged 6 to 11, as it usually takes little time to try to achieve it.

In addition to completely finding the open knight's tour, students can try to create the longest possible one on the given chessboard and compete in this way.

Closed knight's tours are the next step, and starting again with smaller chessboards is advisable.

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